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All hypoenergetic graphs with maximum degree at most 3

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ABSTRACT

The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues. A connected graph G of order n is said to be hypoenergetic if $E(G) < n$. All connected hypoenergetic graphs with maximum degree $\Delta \leqslant 3$ have been characterized. In addition to the four (earlier known) hypoenergetic trees, we now show that complete bipartite graph $K_{2,3}$ is the only hypoenergetic cycle-containing hypoenergetic graph. By this, the validity of a conjecture by Majstorović et al. has been confirmed.

1. Introduction

We use Bondy and Murty [1] for terminology and notations not defined here. Let G be a simple graph with n vertices and m edges. The *cyclomatic number* of a connected graph G is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a *k-cyclic graph*. In particular, for $c(G) = 0, 1, 2$ or 3 we call G a tree, unicyclic, bicyclic or tricyclic graph, respectively. Denote by Δ the maximum degree of a graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix $A(G)$ of G are said to be the eigenvalues of the graph G. The *energy* of G is defined as

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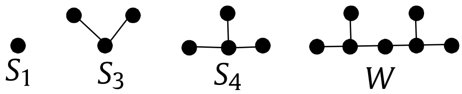

Fig. 1. The hypoenergetic trees with maximum degree at most 3.

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For several classes of graphs it has been demonstrated that the energy exceeds the number of vertices (see, [3]). In 2007, Nikiforov [8] showed that for almost all graphs,

$$E = \left(\frac{4}{3\pi} + o(1)\right) n^{3/2}.$$

Thus the number of graphs satisfying the condition $E < n$ is relatively small. In [5], a connected graph G of order n is called *hypoenergetic* if $E(G) < n$.

Gutman et al. [4] gave results on hypoenergetic trees. You and Liu [10] studied hypoenergetic unicyclic and bicyclic graphs. You, Liu and Gutman [11] considered hypoenergetic tricyclic and k-cyclic graphs. In [6], the present authors showed that there exist hypoenergetic k-cyclic graphs of order n and maximum degree Δ for all (suitable large) n and Δ; And for $\Delta \geqslant 4$ there exist hypoenergetic unicyclic, bicyclic and tricyclic graphs for all n except very few small values of n. For hypoenergetic graphs with $\Delta \leqslant 3$, we have the following results.

Lemma 1.1 [4]. *There exist only four hypoenergetic trees with $\Delta \leqslant 3$, dipicted in Fig. 1.*

Lemma 1.2 [9]. *Let G be a graph of order n with at least n edges and with no isolated vertices. If G is quadrangle-free and $\Delta(G) \leqslant 3$, then $E(G) > n$.*

We will prove the following result in next section.

Theorem 1.3. *Complete bipartite graph $K_{2,3}$ is the only hypoenergetic connected cycle-containing (or cyclic) graph with $\Delta \leqslant 3$.*

Therefore, combining Lemma 1.1 and Theorem 1.3, all connected hypoenergetic graphs with maximum degree at most 3 have been characterized.

Theorem 1.4. *S_1, S_3, S_4, W (see Fig. 1) and $K_{2,3}$ are the only 5 hypoenergetic connected graphs with $\Delta \leqslant 3$.*

By this, the validity of the following conjecture by Majstorović et al. [7] has been confirmed.

Conjecture 1.5 [7]. *Complete bipartite graph $K_{2,3}$ is the only hypoenergetic connected quadrangle-containing graph with $\Delta \leqslant 3$.*

2. Main results

The following two lemmas are need in the sequel.

Lemma 2.1 [6]. *$K_{2,3}$ is the only hypoenergetic graph with $\Delta \leqslant 3$ among all unicyclic and bicyclic graphs.*

Lemma 2.2 [2]. *If F is an edge cut of a simple graph G, then $E(G - F) \leqslant E(G)$, where $G - F$ is the subgraph obtained from G by deleting the edges in F.*

Proof of Theorem 1.3. *Notice that $K_{2,3}$ is hypoenergetic by Lemma 2.1. Let G be a connected cyclic graph with $G \ncong K_{2,3}$, $\Delta \leqslant 3$ and $c(G) = m - n + 1 \geqslant 1$. In the following we show that G is non-hypoenergetic by induction on $c(G)$. It follows from Lemma 2.1 that the result is true if $c(G) \leqslant 2$. We assume that G is non-hypoenergetic for $1 \leqslant c(G) < k$. Now let G be a graph with $c(G) = k \geqslant 3$. In the following we will repeatedly make use of the following claim:*

Claim 1. *If there exists an edge cut F of G such that $G - F$ has exactly two components G_1, G_2 with $0 \leqslant c(G_1), c(G_2) < k$ and $G_1, G_2 \ncong S_1, S_3, S_4, W, K_{2,3}$, then we are done.*

Proof. It follows from Lemma 1.1 and the induction hypothesis that G_1 and G_2 are non-hypoenergetic. By Lemma 2.2, we have $E(G) \geqslant E(G - F)$. Therefore

$$E(G) \geqslant E(G - F) = E(G_1) + E(G_2) \geqslant |V(G_1)| + |V(G_2)| = n,$$

which proves the claim. □

For convenience, we call an edge cut F of G a *good edge cut* if F satisfies the conditions in Claim 1. In what follows, we use $\widehat{G}$ to denote the graph obtained from G by repeatedly deleting the pendent vertices. Clearly, $c(\widehat{G}) = c(G)$. Denote by $\kappa'(\widehat{G})$ the edge connectivity of $\widehat{G}$. Since $\Delta(\widehat{G}) \leqslant 3$, we have $1 \leqslant \kappa'(\widehat{G}) \leqslant 3$. Therefore, we only need to consider the following three cases.

Case 1. $\kappa'(\widehat{G}) = 1$.

Let e be a cut edge of $\widehat{G}$. Then $\widehat{G} - e$ has exactly two components, say, H_1 and H_2. It is clear that $c(H_1) \geqslant 1, c(H_2) \geqslant 1$ and $c(H_1) + c(H_2) = k$. Consequently, $G - e$ has exactly two components G_1 and G_2 with $c(G_1) \geqslant 1, c(G_2) \geqslant 1$ and $c(G_1) + c(G_2) = k$, where H_i is a subgraph of G_i for $i = 1, 2$. If neither G_1 nor G_2 is isomorphic to $K_{2,3}$, then we are done by Claim 1. Otherwise, by symmetry we assume that $G_1 \cong K_{2,3}$. Then G must have the structure as given in Fig. 2a. Now, let $F = \{e_1, e_2\}$. Then $G - F$ has exactly two components G_1' and G_2', where G_1' is a quadrangle and G_2' is a graph obtained from G_2 by adding a pendent edge. Therefore we have that $c(G_2') = k - 2$ and $G_2' \ncong K_{2,3}$, and so we are done by Claim 1.

Case 2. $\kappa'(\widehat{G}) = 2$.

Let $F = \{e_1, e_2\}$ be an edge cut of $\widehat{G}$. Then $\widehat{G} - F$ has exactly two components, say, H_1 and H_2. Clearly, $c(H_1) + c(H_2) = k - 1 \geqslant 2$.

Subcase 2.1. $c(H_1) \geqslant 1$ and $c(H_2) \geqslant 1$.

Therefore, $G - F$ has exactly two components G_1 and G_2 with $c(G_1) \geqslant 1$, $c(G_2) \geqslant 1$ and $c(G_1) + c(G_2) = k - 1$, where H_i is a subgraph of G_i for $i = 1, 2$. If neither G_1 nor G_2 is isomorphic to $K_{2,3}$, then we are done by Claim 1. Otherwise, by symmetry we assume that $G_1 \cong K_{2,3}$. Then G must have the structure as given in Fig. 2b. Now, let $F' = \{e_2, e_3, e_4\}$. Then it is easy to see that F' is a good edge cut. The proof is thus complete.

Subcase 2.2. One of H_1 and H_2, say H_2 is a tree.

Therefore, $G - F$ has exactly two components G_1 and G_2 with $c(G_1) = k - 1$ and $c(G_2) = 0$, where H_i is a subgraph of G_i for $i = 1, 2$. If $G_1 \ncong K_{2,3}$ and $G_2 \ncong S_1, S_3, S_4, W$, then we are done by Claim 1. So we assume that this is not true. We only need to consider the following five cases.

Subsubcase 2.2.1. $G_2 \cong S_1$.

Let $V(G_2) = \{x\}, e_1 = xx_1$ and $e_2 = xx_2$. It is clear that $d_{G_1}(x_2) = 1$ or 2. If $d_{G_1}(x_2) = 1$, let $N_{G_1}(x_2) = \{y_1\}$ (see Fig. 3a, where y_1 may be equal to x_1). Let $F' = \{e_1, x_2y_1\}$. Then $G - F'$ has exactly two components G_1' and G_2', where G_1' is a graph obtained from G_1 by deleting a pendent vertex and G_2' is a

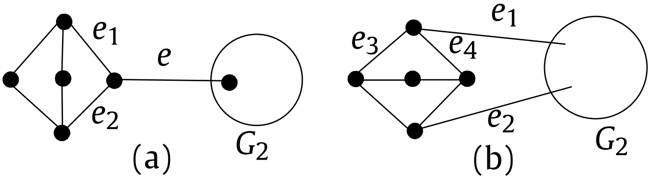

Fig. 2. The graphs in Case 1 and Subcase 2.1 of Theorem 1.4.

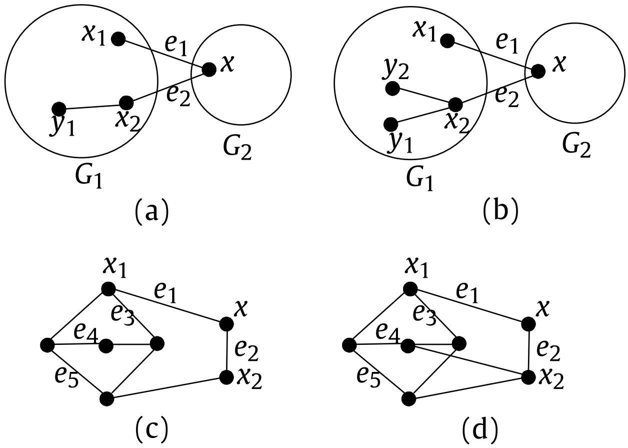

Fig. 3. The graphs in Subsubcase 2.2.1 of Theorem 1.4.

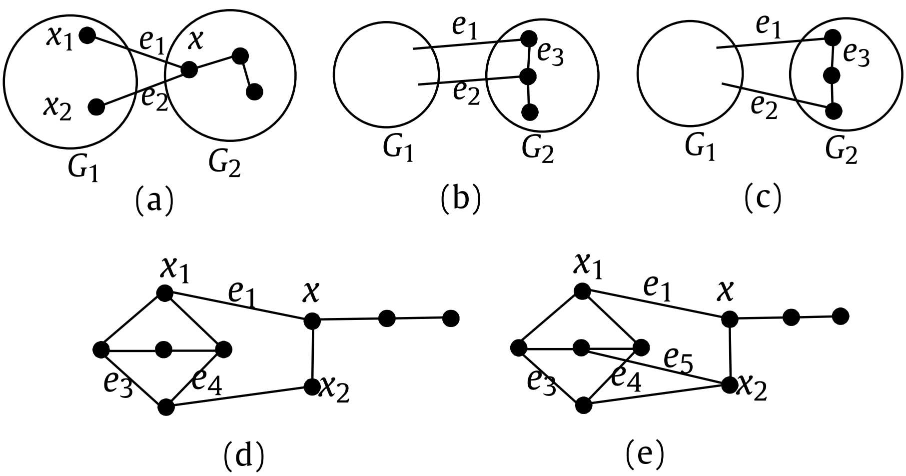

Fig. 4. The graphs in Subsubcase 2.2.2 of Theorem 1.4.

tree of order 2. Therefore, $c(G_1') = k - 1$. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise, G must be the graph as given in Fig. 3c. It is easy to see that $F'' = \{e_1, e_3, e_4, e_5\}$ is a good edge cut.

If $d_{G_1}(x_2) = 2$, let $N_{G_1}(x_2) = \{y_1, y_2\}$ (see Fig. 3b, where one of y_1 and y_2 may be equal to x_1). Let $F' = \{e_1, x_2y_1, x_2y_2\}$. Then $G - F'$ has exactly two components G_1' and G_2' such that G_1' is a graph obtained from G_1 by deleting a vertex of degree 2 and G_2' is a tree of order 2. Therefore, $c(G_1') = k - 2$. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise, G must be the graph as given in Fig. 3d. It is easy to see that $F'' = \{e_1, e_3, e_4, e_5\}$ is a good edge cut.

Subsubcase 2.2.2. $G_2 \cong S_3$.

If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Fig. 4a. Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F'$ has exactly two components G_1' and G_2' satisfying that $c(G_1') = k - 1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k - 2$ if $d_{G_1}(x_2) = 2$ and G_2' is a path of order 4. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise G must be the graph as given in Fig. 4d or e. In the former case $F'' = \{e_1, e_3, e_4\}$ is a good edge cut while in the latter case $F'' = \{e_1, e_3, e_4, e_5\}$ is a good edge cut.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Fig. 4b or c. It is easy to see that $F' = \{e_2, e_3\}$ is a good edge cut. The proof is thus complete.

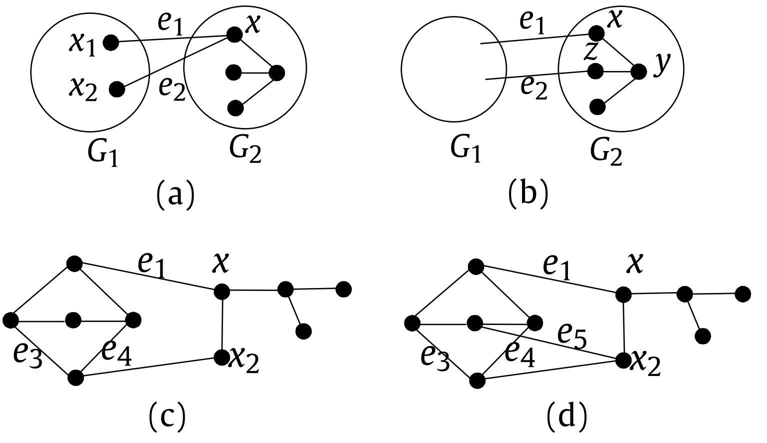

Fig. 5. The graphs in Subsubcase 2.2.3 of Theorem 1.4.

Subsubcase 2.2.3. $G_2 \cong S_4$.

If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Fig. 5a. Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F'$ has exactly two components G_1' and G_2' satisfying that $c(G_1') = k - 1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k - 2$ if $d_{G_1}(x_2) = 2$ and G_2' is a tree of order 5. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise G is the graph as given in Fig. 5c or d. In the former case $F'' = \{e_1, e_3, e_4\}$ is a good edge cut while in the latter case $F'' = \{e_1, e_3, e_4, e_5\}$ is a good edge cut.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Fig. 5b. It is easy to see that $F' = \{xy, yz\}$ is a good edge cut. The proof is thus complete.

Subsubcase 2.2.4. $G_2 \cong W$.

If e_1, e_2 are incident with a common vertex in G_2, then G must have the structure as given in Fig. 6a. Similar to the proof of Subsubcase 2.2.1, we can obtain that there exists an edge cut F' such that $G - F'$ has exactly two components G_1' and G_2' satisfying that $c(G_1') = k - 1$ if $d_{G_1}(x_2) = 1$ or $c(G_1') = k - 2$ if $d_{G_1}(x_2) = 2$ and G_2' is a tree of order 8. If $G_1' \ncong K_{2,3}$, then we are done by Claim 1. Otherwise, G is the graph as given in Fig. 6e or f. In the former case $F'' = \{e_1, e_3, e_4\}$ is a good edge cut while in the latter case $F'' = \{e_1, e_3, e_4, e_5\}$ is a good edge cut.

If e_1, e_2 are incident with two different vertices in G_2, then G must have the structure as given in Fig. 6b, c or d. It is easy to see that $F' = \{xy, yz\}$ is a good edge cut. The proof is thus complete.

Subsubcase 2.2.5. $G_1 \cong K_{2,3}$ and $G_2 \ncong S_1, S_3, S_4, W$.

It is easy to see that G must have the structure as given in Fig. 7a or b. Let $F' = \{e_2, e_3, e_4\}$. Then $G - F'$ has exactly two components G_1' and G_2', where G_1' is a quadrangle and G_2' is obtained from G_2 by adding a pendent edge. If $G_2' \ncong S_1, S_3, S_4, W$, then we are done by Claim 1. Otherwise, since $\Delta(G) \leqslant 3$ and $G_2 \ncong S_1, S_3, S_4, W$, G must be isomorphic to the graph as given in Fig. 3c or Fig. 7c, d, e or f. In the first case we are done while in the other cases $F'' = \{e_1, e_4, e_5, e_6\}$ is a good edge cut. The proof is thus complete.

Case 3. $\kappa'(\widehat{G}) = 3$.

Noticing that $\Delta(\hat{G}) \leqslant 3$ and $\Delta(G) \leqslant 3$, we obtain that $G = \hat{G}$ is a connected 3-regular graph.

Let $F = \{e_1, e_2, e_3\}$ be an edge cut of G. Then $G - F$ has exactly two components, say, G_1 and G_2. Clearly, $c(G_1) + c(G_2) = k - 2 \geqslant 1$.

Subcase 3.1. $c(G_1) \geqslant 1$ and $c(G_2) \geqslant 1$.

If neither G_1 nor G_2 is isomorphic to $K_{2,3}$, then we are done by Claim 1. Otherwise, by symmetry we assume that $G_1 \cong K_{2,3}$. Then G must have the structure as given in Fig. 8a. Let $F' = \{e_1, e_2, e_4, e_5\}$. Then it is easy to see that F' is a good edge cut. The proof is thus complete.

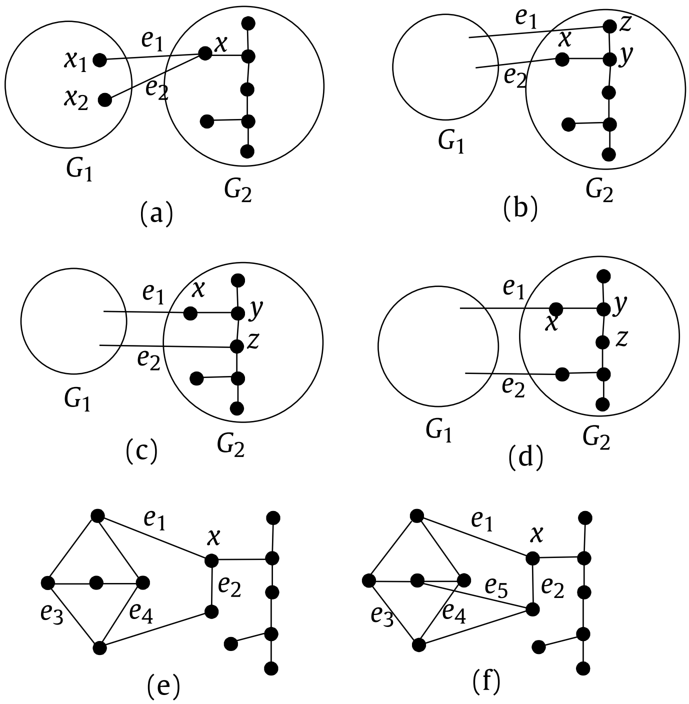

Fig. 6. The graphs in Subsubcase 2.2.4 of Theorem 1.4.

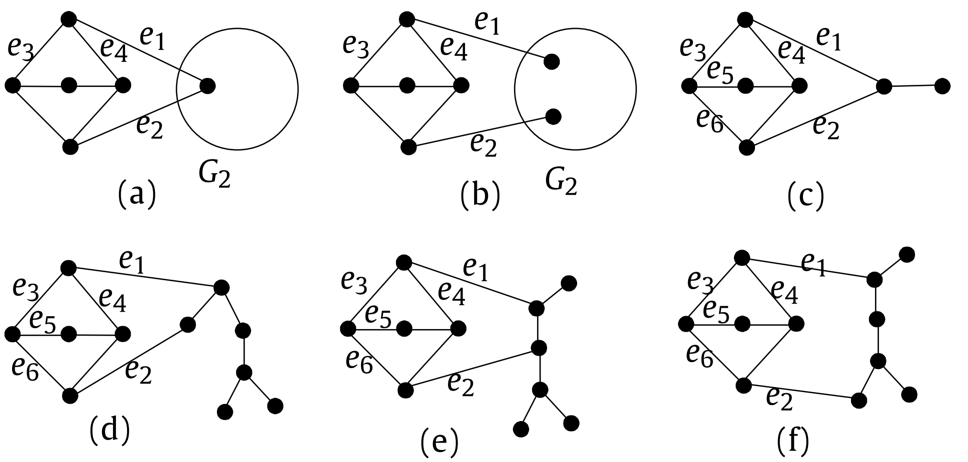

Fig. 7. The graphs in Subsubcase 2.2.5 of Theorem 1.4.

Subcase 3.2. One of G_1 and G_2, say G_2 is a tree.

Let $|V(G_2)| = n_2$. Then we have $3n_2 = \sum_{v \in V(G_2)} d_G(v) = 2(n_2 - 1) + 3 = 2n_2 + 1$. Therefore, $n_2 = 1$, i.e., $G_2 = S_1$. Let $V(G_2) = \{x\}$, $e_1 = xx_1$, $e_2 = xx_2$ and $e_3 = xx_3$. Let $N_{G_1}(x_2) = \{y_1, y_2\}$ (see Fig. 8b). Let $F' = \{e_1, e_3, x_2y_1, x_2y_2\}$. Then $G - F'$ has exactly two components G_1' and G_2', where G_1' is a graph obtained from G_1 by deleting a vertex of degree 2 and G_2' is a tree of order 2. Therefore,

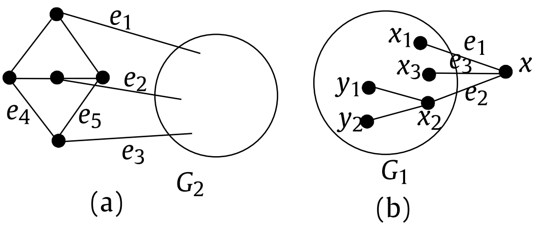

Fig. 8. The graphs in Case 3 of Theorem 1.4.

$c(G_1') = k - 3$. It is easy to check that $G_1' \ncong K_{2,3}$. If G_1' is a tree, then we have $|V(G_1')| = 2$, since $3|V(G_1')| = \sum_{v \in V(G_1')} d_G(v) = 2(|V(G_1')| - 1) + 4 = 2|V(G_1')| + 2$. Therefore, we are done by Claim 1. $\square$

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, Elsevier, London, New York, 1976.
[2] J. Day, W. So, Graph energy change due to edge deletion, Linear Algebra Appl. 428 (2008) 2070–2078.
[3] I. Gutman, On graphs whose energy exceeds the number of vertices, Linear Algebra Appl. 429 (2008) 2670–2677.
[4] I. Gutman, X. Li, Y. Shi, J. Zhang, Hypoenergetic trees, Match 60 (2009) 415–426.
[5] I. Gutman, S. Radenković, Hypoenergetic molecular graphs, Indian J. Chem. 46A (2007) 1733–1736.
[6] X. Li, H. Ma, Hypoenergetic and strongly hypoenergetic *k*-cyclic graphs, Match, accepted for publication.
[7] S. Majstorović, A. Klobučar, I. Gutman, Selected topics from the theory of graph energy: hpoenergetic graphs, in: Applications of Graph Spectra, Math. Inst., Belgrade, 2009, pp. 65–105.
[8] V. Nikiforov, Graphs and matrices with maximal energy, J. Math. Anal. Appl. 327 (2007) 735–738.
[9] V. Nikiforov, The energy of C_4-free graphs of bounded degree, Linear Algebra Appl. 428 (2008) 2569–2573.
[10] Z. You, B. Liu, On hypoenergetic unicyclic and bicyclic graphs, Match 61 (2009) 479–486.
[11] Z. You, B. Liu, I. Gutman, Note on hypoenergetic graphs, Match 62 (2009) 491–498.